Fast MRF Optimization with Application to Depth Reconstruction
Supplementary Material

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In the supplementary material, we provide more information on some proofs concerning the lower envelope and the algorithms for its construction.

1. Lower Envelope Complexity

Here we provide proofs concerning the number of intersections between pairs of shifted penalty functions. For simplicity of exposition, we restrict ourselves here to strictly differentiable penalties. The proof for the L^1 penalty, which is neither strictly convex nor differentiable, is trivial.

Lemma 1.1. *If $f(x) : \mathbb{R} \to \mathbb{R}$ is a strictly convex differentiable function and $g(x) = f(x + a) + b$ where $a \neq 0$, then f and g intersect at most once.*

Proof. We prove by contradiction. Suppose f and g intersect at x_0 and x_1 ($x_0 < x_1$). Let $\Delta = x_1 - x_0$. Then

$$\begin{aligned} f(x_1) - f(x_0) &= g(x_1) - g(x_0) \implies \\ f(x_1) - f(x_0) &= f(x_1 + a) - f(x_0 + a) \implies \\ \frac{f(x_1) - f(x_0)}{\Delta} &= \frac{f(x_1 + a) - f(x_0 + a)}{\Delta} \implies \\ h(x_0) &= h(x_0 + a), \end{aligned}$$

where $h(x) = \frac{f(x+\Delta)-f(x)}{\Delta}$ is an increasing function because f' is increasing and

$$h'(x) = \frac{f'(x + \Delta) - f'(x)}{\Delta} > 0.$$

Therefore $h(x_0) = h(x_0 + a)$ implies that $a = 0$. This contradicts our assumption that $a \neq 0$. □

Lemma 1.2. *If $f(x) : \mathbb{R} \to \mathbb{R}$ is a strictly concave differentiable function and $g(x) = f(x + a) + b$ where $a \neq 0$, then f and g intersect at most once.*

Proof. The negated functions $-f$ and $-g$ are strictly convex differentiable functions. According to Lemma 1.1, $-f$ and $-g$ intersect at most once. Therefore f and g intersect at most once. □

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Lemma 1.3. *If $f(x) : \mathbb{R} \to \mathbb{R}$ is a strictly convex differentiable function and $g(x) : \mathbb{R} \to \mathbb{R}$ is a strictly concave function, then f and g intersect at most twice.*

Proof. Let $h(x) = f(x) - g(x)$ and h is convex. $f(x) = g(x)$ is equivalent to $h(x) = 0$. There are at most two solutions for $h(x) = 0$ since h is convex. □

Lemma 1.4. *If $f(x) = log(1 + \frac{x^2}{2\epsilon^2})$ is the Lorentzian penalty and $g(x) = f(x + a) + b$ where $a \neq 0$, then f and g intersect at most twice.*

Proof. We solve $f(x) = g(x)$ for x,

$$\begin{aligned} log(1 + \frac{x^2}{2\epsilon^2}) &= log(1 + \frac{(x + a)^2}{2\epsilon^2}) + b \implies \\ 1 + \frac{x^2}{2\epsilon^2} &= e^b(1 + \frac{(x + a)^2}{2\epsilon^2}) \implies \\ 0 &= \frac{e^b - 1}{2\epsilon^2}x^2 + \frac{ae^b}{\epsilon^2}x + \frac{a^2e^b}{2\epsilon^2} + e^b - 1. \end{aligned}$$

If $b \neq 0$, this is a quadratic equation and there are at most two solutions for x and f intersects g at most twice. If $b = 0$, this is a linear equation and f intersects g at most once. □

Lemma 1.5. *If $f(x) = (x^2 + \epsilon^2)^\alpha$ is the generalized Charbonnier penalty where $\alpha \in (0, \frac{1}{2})$ and $g(x) = f(x + a) + b$ where $a \neq 0$, then f and g intersect at most twice.*

Proof. Without loss of generality, we assume that $a < 0$ and $b < 0$. If $a > 0$, we reflect g and f over the y axis; if $b > 0$, we reflect g and f over the x axis. If $b = 0$, $f(x) = g(x)$ can be reduced to $x = -\frac{a}{2}$ and there is only one intersection.

We observe that $\lim_{x\to-\infty} f(x) - g(x) = -b > 0$ and $\lim_{x\to+\infty} f(x) - g(x) = -b > 0$. f is above g at $\pm\infty$. This implies that f and g can only intersect an even number of times. Thus we only need to show that f and g intersect at most three times.

f is a piecewise convex and concave function. The second derivative is

$$f''(x) = 2\alpha \left(\epsilon^2 + x^2\right)^{\alpha-2} \left(\epsilon^2 + (2\alpha - 1)x^2\right),$$

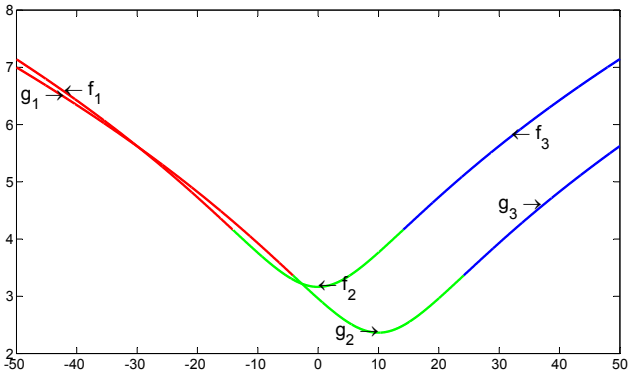

Figure 1. $f(x) = (x^2 + \epsilon^2)^\alpha$ where $\epsilon = 10$ and $\alpha = 0.25$. $g(x) = f(x + a) + b$ where $a = -10$ and $b = -0.8$. The red curves are f_1 and g_1, the green curves are f_2 and g_2, and the blue curves are f_3 and g_3.

so $f''(x) > 0$ iff $x \in (-t, t)$, where $t = \frac{\epsilon}{\sqrt{1-2\alpha}}$. Thus f is convex on $(-t, t)$ and concave on $(-\infty, -t]$ and $[t, +\infty)$. Similarly, g is convex on $(-t - a, t - a)$ and concave on $[-\infty, -t - a]$ and $[t - a, +\infty)$.

We analyze how g intersects f in terms of their three convex or concave parts. See Figure 1 for illustration. We denote $f_1 = \{f(x) : x \leq -t\}$ (concave), $f_2 = \{f(x) : -t < x < t\}$ (convex), $f_3 = \{f(x) : x \geq t\}$ (concave), $g_1 = \{g(x) : x \leq -t - a\}$ (concave), $g_2 = \{g(x) : -t - a < x < t - a\}$ (convex), and $g_3 = \{g(x) : x \geq t - a\}$ (concave). The proof proceeds by the following three claims:

- g_1 intersects f at most twice. To show this, consider two cases:
  - g_1 intersects f_3. In this case, g_1 intersects f_3 once because g_1 is decreasing and f_3 is increasing. g_1 does not intersect f_2 because g_1 intersecting f_3 implies $g_1(t) > f_3(t)$ and $g_1(x) > g_1(t) > f_3(t) > f_2(x)$ for any x defined on both g_1 and f_2. g_1 intersects f_1 at most once by Lemma 1.2.
  - g_1 does not intersect f_3. In this case, g_1 intersects f_1 at most once by Lemma 1.2. g_1 intersects f_2 at most twice by Lemma 1.3. However, it is impossible that g_1 intersects f_2 twice while g_1 intersects f_1 once, because g_1 intersecting f_2 twice implies that $g_1(-t) < f_1(-t)$ and g_1 intersecting f_1 once implies $g_1(-t) > f_1(-t)$ (recall that $\lim_{x\to-\infty} g(x) - f(x) < 0$).
- g_2 intersects f at most once. g_2 intersects f_2 at most once by Lemma 1.1. g_2 does not intersect f_1 because the domains of g_2 and f_1 do not overlap. g_2 does not intersect f_3 because $g_2(x) < g(t - a) = f(t) + b < f(t) \leq f_3(x)$ for any x defined on both g_2 and f_3.

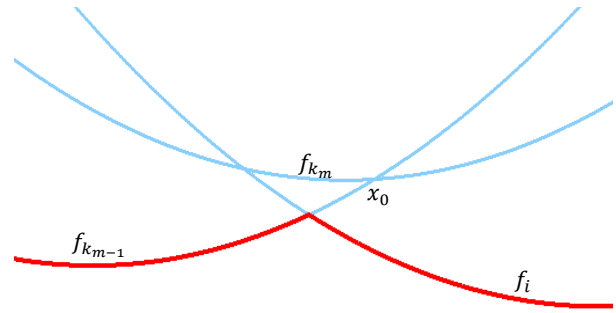

Figure 2. f_{k_m} is above the lower envelope as $f_i(x_0) \leq f_{k_m}(x_0)$ and should be removed from S.

- g_3 does not intersect f. g_3 does not intersect f_1 and f_2 because the domains of g_3, f_1 and f_2 do not overlap. g_3 does not intersect f_3 because $g_3(x) = f_3(x + a) + b < f_3(x + a) < f_3(x)$ for any x defined on g_3.

In summary, f and g intersect at most three times. Since f and g can only intersect an even number of times, f intersects g at most twice. □

2. Lower Envelope Construction

Let e be the lower envelope of the functions $\{f_0, \ldots, f_{D-1}\}$:

$$\begin{aligned} f_k(d) &= \mathbf{C}(x - 1, y, k) + \rho(d - k), \\ e(d) &= \min_{0 \leq k < D} f_k(d), \end{aligned}$$

where ρ is a convex function.

If f_j intersects f_k ($j < k$), by Lemma 1.1, $f_j(d) < f_k(d)$ iff $d < \Lambda(f_j, f_k)$ where $\Lambda(f_j, f_k)$ is the intersection point between f_j and f_k. This allows us to represent e by a subset of functions $S = \{f_{k_1}, \ldots, f_{k_m}\}$ ($k_i < k_{i+1}$):

$$e(d) = f_{k_i}(d),$$

where $d \in [\Lambda(f_{k_{i-1}}, f_{k_i}), \Lambda(f_{k_i}, f_{k_{i+1}})]$. Each function in S contributes a single contiguous curve to the lower envelope and functions above the lower envelope are not in S.

The lower envelope can be constructed by incrementally adding f_i. Suppose we have computed the lower envelope of $\{f_0, \ldots, f_{i-1}\}$, represented by $S = \{f_{k_1}, \ldots, f_{k_m}\}$. Now we compute the lower envelope of $\{f_0, \ldots, f_i\}$ by updating S:

1. If f_i is totally above f_{k_m}, then ignore f_i and S remains the same.

2. Otherwise, f_i will contribute to the lower envelope, as illustrated in Figure 2. Let $x_0 = \Lambda(f_{k_{m-1}}, f_{k_m})$. If $f_i(x_0) \leq f_{k_m}(x_0)$, f_{k_m} will be totally above the lower envelope after adding f_i and we will remove f_{k_m} from S. We keep removing the last function in S if it is above the lower envelope after adding f_i and then we will add f_i to S.

Each function can only be added once and removed once. The algorithm thus runs in time $O(D)$ assuming Λ can be evaluated in constant time.

Truncated functions. For truncated penalty functions $\min\{\rho, T\}$, we can derive the lower envelope e' as

$$\begin{aligned} e'(d) &= \min(e(d), T'), \\ T' &= T + \min_d \mathbf{C}(x-1, y, d). \end{aligned}$$

Thus for truncated penalties we can simply construct the lower envelope of the unrestricted penalties and then cut it with a single horizontal line. This introduces at most a linear number of cuts, which can be computed in linear time by sweeping the lower envelope e.