Efficient Nonlocal Regularization for Optical Flow – Supplementary material

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1 Exponential mixture approximation

Here we briefly describe how to fit an exponential mixture

$$\mu_{\boldsymbol{\omega},\boldsymbol{\sigma}}(x) = T - \sum_{n=1}^{K} \omega_n \exp\left(-\frac{x^2}{2\sigma_n^2}\right) \tag{1}$$

to an arbitrary truncated penalty function $\tilde{\rho}(x) = \min(T, \rho(x))$. We pose the fitting problem as a least squares optimization

$$\begin{aligned} E(\boldsymbol{\omega},\boldsymbol{\sigma}) &= \int_{-\infty}^{\infty} \left(\mu_{\boldsymbol{\omega},\boldsymbol{\sigma}}(x) - \tilde{\rho}(x)\right)^2 dx \\ &= \int_{-\infty}^{\infty} \left(T - \sum_{n=1}^{K} \omega_n \exp\left(-\frac{x^2}{2\sigma_n^2}\right) - \min(T,\rho(x))\right)^2 dx \\ &= \int_{-\infty}^{\infty} \left(\max(0, T-\rho(x)) - \sum_{n=1}^{K} \omega_n \exp\left(-\frac{x^2}{2\sigma_n^2}\right)\right)^2 dx. \end{aligned} \tag{2}$$

It easy to see that this fitting problem is non-convex and has multiple minima, one for each permutation of the parameters $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$. To disambiguate those minima we iteratively fit a K-kernel model using the solution of the $(K-1)$-kernel model as an initialization. The new kernel is initialized with a standard deviation of 0.1 times the smallest standard deviation present in the mixture model. The weight is computed from the standard deviation.

For the weights $\boldsymbol{\omega}$, Equation 2 describes a linear least squares problem, while for $\boldsymbol{\sigma}$ it is non-linear. We solve both problems separately and iterate between finding $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$.

We start by minimizing Equation 2 with respect to $\boldsymbol{\omega}$:

$$\begin{aligned} \frac{\partial}{\partial \omega_m} E(\boldsymbol{\omega},\boldsymbol{\sigma}) &= 2\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_m^2}\right)\left(\sum_{n=1}^{K} \omega_n \exp\left(-\frac{x^2}{2\sigma_n^2}\right) - \max(0,T-\rho(x))\right) dx \\ &= 2\int_{-\infty}^{\infty} \sum_{n=1}^{K} \omega_n \exp\left(-\frac{x^2}{2\sigma_m^2}\right)\exp\left(-\frac{x^2}{2\sigma_n^2}\right) dx \\ &\quad - 2\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_m^2}\right)\max(0,T-\rho(x)) dx \\ &= 2\sum_{n=1}^{K} \omega_n \frac{\sqrt{2\pi}\sigma_n\sigma_m}{\sqrt{\sigma_n^2+\sigma_m^2}} - 2\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_m^2}\right)\max(0,T-\rho(x)) dx \\ &= 0 \end{aligned}$$

This defines a linear system, which can be solved using a standard solver. We evaluate the integral $\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma_m^2}\right)\max(0,T-\rho(x))dx$ using numeric integration on the range $[-10\sigma_n, 10\sigma_n]$ [1].

Equation 2 describes a non-linear least squares optimization with respect to $\boldsymbol{\sigma}$. We first linearize the exponential mixture using a first-order Taylor approximation

$$\begin{aligned} \mu_{\boldsymbol{\omega},\boldsymbol{\sigma}}(x) &\approx \mu_{\boldsymbol{\omega},\boldsymbol{\sigma}^0}(x) + \Delta\boldsymbol{\sigma}\nabla_{\boldsymbol{\sigma}}\mu_{\boldsymbol{\omega},\boldsymbol{\sigma}^0}(x) \\ &\approx T - \sum_{n=1}^{K} \omega_n \left(1 + \frac{\Delta\sigma_i x^2}{{\sigma_n^0}^3}\right) \exp\left(-\frac{x^2}{2{\sigma_n^0}^2}\right) \end{aligned}$$

Next we substitute the Taylor expansion into our least-squares objective and find $\Delta\boldsymbol{\sigma}$.

$$\begin{aligned} \frac{\partial}{\partial \Delta\sigma_m} E(\boldsymbol{\omega},\boldsymbol{\sigma}) =& 2\int_{-\infty}^{\infty} \frac{x^2\omega_m}{{\sigma_m^0}^3} \exp\left(-\frac{x^2}{2{\sigma_m^0}^2}\right) \left(\sum_{n=1}^{K} \omega_n \left(1 + \frac{\Delta\sigma_i x^2}{{\sigma_n^0}^3}\right) \exp\left(-\frac{x^2}{2{\sigma_n^0}^2}\right) \right. \\ & \left. - \max(0, T - \rho(x)) \right) dx \\ =& \frac{2\omega_m}{{\sigma_m^0}^3} \sum_{n=1}^{K} \omega_n \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2{\sigma_m^0}^2}\right) \left(1 + \frac{\Delta\sigma_i x^2}{{\sigma_n^0}^3}\right) \exp\left(-\frac{x^2}{2{\sigma_n^0}^2}\right) \\ & - \frac{2\omega_m}{{\sigma_m^0}^3} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2{\sigma_m^0}^2}\right) \max(0, T-\rho(x)) dx \\ =& \frac{2\omega_m}{{\sigma_m^0}^3} \sum_{n=1}^{K} \omega_n \left(\frac{\sqrt{2\pi}{\sigma_n^0}^3{\sigma_m^0}^3}{({\sigma_n^0}^2 + {\sigma_m^0}^2)^{\frac{3}{2}}} + \frac{\Delta\sigma_i}{{\sigma_n^0}^3} \frac{3\sqrt{2\pi}{\sigma_n^0}^5{\sigma_m^0}^5}{({\sigma_n^0}^2 + {\sigma_m^0}^2)^{\frac{5}{2}}} \right) \\ & - \frac{2\omega_m}{{\sigma_m^0}^3} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2{\sigma_m^0}^2}\right) \max(0, T-\rho(x)) dx \\ =& 2\omega_m \sum_{n=1}^{K} \omega_n \left(\frac{\sqrt{2\pi}{\sigma_n^0}^3}{({\sigma_n^0}^2 + {\sigma_m^0}^2)^{\frac{3}{2}}} + \Delta\sigma_i \frac{3\sqrt{2\pi}{\sigma_m^0}^2{\sigma_n^0}^2}{({\sigma_n^0}^2 + {\sigma_m^0}^2)^{\frac{5}{2}}} \right) \\ & - \frac{2\omega_m}{{\sigma_m^0}^3} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2{\sigma_m^0}^2}\right) \max(0, T-\rho(x)) dx \\ =& 0. \end{aligned}$$

We again evaluate the integral using numeric integration.

The remaining linear system can be solved using a standard method. To guarantee convergence we add a Marquardt coefficient λ by multiplying the diagonal of the linear system by $(1+\lambda)$. A large value λ makes the problem diagonally dominant and ensures convergence. The resulting algorithm is equivalent to the Levenberg-Marquardt algorithm over a continuous domain [1].

References

1. W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes: The Art of Scientific Computing.* Cambridge University Press, 3rd edition, 2007.